

## DYNAMICS IN THE COMPLEX BIDISC

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ABSTRACT. Let  $\Delta^n$  be the unit polydisc in  $\mathbb{C}^n$  and let  $f$  be a holomorphic self map of  $\Delta^n$ . When  $n = 1$ , it is well known, by Schwarz's lemma, that  $f$  has at most one fixed point in the unit disc. If no such point exists then  $f$  has a unique boundary point, call it  $x \in \partial\Delta$ , such that every horocycle  $E(x, R)$  of center  $x$  and radius  $R > 0$  is sent into itself by  $f$ . This boundary point is called the *Wolff point* of  $f$ . In this paper we propose a definition of Wolff points for holomorphic maps defined on a bounded domain of  $\mathbb{C}^n$ . In particular we characterize the set of Wolff points,  $W(f)$ , of a holomorphic self-map  $f$  of the bidisc in terms of the properties of the components of the map  $f$  itself.

## 1. INTRODUCTION

Let  $D$  be a bounded domain of  $\mathbb{C}^n$  and let  $f$  be a holomorphic self map of  $D$ . We denote by  $k_D$  the Kobayashi distance on  $D$  [8], [9],[5] and, as in [1], [3], we define the *small horosphere*  $E(x, R)$  and the *big horosphere*  $F(x, R)$  of center  $x$  and radius  $R$  as follows:

$$(1.1) \quad \begin{aligned} E(x, R) &= \{z \in D : \limsup_{w \rightarrow x} [k_D(z, w) - k_D(0, w)] < \tfrac{1}{2} \log R\}, \\ F(x, R) &= \{z \in D : \liminf_{w \rightarrow x} [k_D(z, w) - k_D(0, w)] < \tfrac{1}{2} \log R\}. \end{aligned}$$

We say that  $\tau \in \partial D$  is a *Wolff point* of  $f$  if  $f(E(\tau, R)) \subset E(\tau, R)$ , for all  $R > 0$ . Denote by  $W(f)$  the set of *Wolff points* of  $f$  and denote by  $T(f)$  the *target set*

$$T(f) := \{x \in \bar{D} \mid \exists \{k_n \in \mathbb{N}\}, z \in D \text{ such that } f^{k_n}(z) \rightarrow x \text{ as } n \rightarrow \infty\}.$$

If  $D = \Delta = \{z \in \mathbb{C} : |z| < 1\}$ , it is well known, by Schwarz's lemma, that a holomorphic map  $f : \Delta \rightarrow \Delta$  has at most one fixed point in  $\Delta$ . If  $f$  has a fixed point in  $\Delta$ , say  $z_0$ , then  $W(f) = \emptyset$  and  $T(f) = \{z_0\}$ . If  $f$  has no fixed points in  $\Delta$  then, by Wolff's lemma [13] [11],  $W(f)$  is reduced to a boundary point  $x \in \partial\Delta$  and, by the classical Denjoy theorem,  $T(f) = W(f)$ . The same holds for self-maps of  $\mathbb{B}^n$  with no interior fixed points [6], [7], [10] and in particular if  $f$  is a holomorphic self-map of a strongly convex domain  $D$  with  $C^3$ -boundary [2]. On the other hand, if the map has fixed points in  $D$  then either  $W(f) = \emptyset$  and  $T(f)$  is a unique point in  $D$ , or  $T(f)$  is a complex open subvariety  $\Gamma$  of  $D$  (affine in case  $D = \mathbb{B}^n$ ) and  $W(f) = \partial D \cup \bar{\Gamma}$ . In this paper we examine another type of convex domains (not strongly convex, not even with regular boundary): the polydiscs. To avoid technical complications we restrict ourselves to dimension two, so we begin studying the case of the bidisc. In this case, we characterize Wolff points of a holomorphic map  $f : \Delta^2 \rightarrow \Delta^2$  in terms of the properties of the components of the map  $f$ . Thus let  $f : \Delta^2 \rightarrow \Delta^2$  be a holomorphic self-map in the complex bidisc without fixed points in  $\Delta^2$ . Then  $f(x, y) = (f_1(x, y), f_2(x, y))$  with  $f_1, f_2 : \Delta^2 \rightarrow \Delta$  holomorphic functions in  $x$  and  $y$ . Then one of the two following possibilities holds ([6]):

- i) there exists Wolff point of  $f_1(\cdot, y)$ ,  $e^{i\theta_1}$ , which does not depend on  $y$  or

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- ii) there exists a holomorphic function  $F_1 : \Delta \rightarrow \Delta$ , such that  $f_1(F_1(y), y) = F_1(y)$ .  
In this case  $f_1(x, y) = x \Rightarrow x = F_1(y)$ .

Note that if  $f \neq id_\Delta$  then cases *i*), *ii*) exclude each other. Motivated by this result we make the following definition:

**Definition 1.** Let  $f : \Delta^2 \rightarrow \Delta^2$  be a holomorphic function and let  $f_1, f_2$  be its components. The map  $f$  is called of:

- i) *first type* if:
  - there exists a holomorphic function  $F_1 : \Delta \rightarrow \Delta$ , such that  $f_1(F_1(y), y) = F_1(y)$  and
  - there exists a holomorphic function  $F_2 : \Delta \rightarrow \Delta$ , such that  $f_2(x, F_2(x)) = F_2(x)$ .
- ii) *second type* if (up to interchange  $f_1$  with  $f_2$ ):
  - there exists a Wolff point of  $f_1(\cdot, y)$ ,  $e^{i\theta_1}$ , (necessarily independent of  $y$ ) and
  - there exists a holomorphic function  $F_2 : \Delta \rightarrow \Delta$ , such that  $f_2(x, F_2(x)) = F_2(x)$ .
- iii) *third type* if:
  - there exists a Wolff point of  $f_1(\cdot, y)$ ,  $e^{i\theta_1}$ , (independent of  $y$ ) and
  - there exists a Wolff point of  $f_2(x, \cdot)$ ,  $e^{i\theta_2}$ , (independent of  $x$ ).

In case  $f$  is of *first type* and without interior fixed points  $F_1 \circ F_2$  (respectively  $F_2 \circ F_1$ ) must have Wolff point (see Lemma 5). Let  $e^{i\theta_1}$  (respectively by  $e^{i\theta_2}$ ) be the Wolff point of  $F_1 \circ F_2$  (respectively  $F_2 \circ F_1$ .) Let  $\lambda_1 := \lim_{y \rightarrow e^{i\theta_2}} F_1'(y)$  and  $\lambda_2 := \lim_{x \rightarrow e^{i\theta_1}} F_2'(x)$  be respectively the *boundary dilatation coefficients* of  $F_1$  at  $e^{i\theta_2}$  and of  $F_2$  at  $e^{i\theta_1}$  (see Lemma 5). If  $f$  is of *second type* we denote by  $\lambda_2$  the boundary dilatation coefficient of  $F_2$  at  $e^{i\theta_1}$ . Finally we let  $\pi_j$  be the projection on the  $j$ -component. With this notation our main result is:

**Theorem 2.** Let  $f = (f_1, f_2)$  be a holomorphic map, without fixed points in the complex bidisc. If  $f_1 \neq \pi_1$  and  $f_2 \neq \pi_2$ , then there are the following six cases:

- i)  $W(f) = \emptyset$  if and only if  $f$  is of *first type* and  $\lambda_i > 1$  for one  $i = 1, 2$ .
  - ii)  $W(f) = (e^{i\theta_1}, e^{i\theta_2})$  iff  $f$  is of *first type*  $\lambda_i \leq 1$  for each  $i = 1, 2$ .
  - iii)  $W(f) = \{\{e^{i\theta_1}\} \times \Delta\} \cup \{(e^{i\theta_1}, e^{i\theta_2})\}$  iff  $f$  is of *second type* and  $\lambda_2 \leq 1$ .
  - iv)  $W(f) = \{\{e^{i\theta_1}\} \times \Delta\}$  iff  $f$  is of *second type* and  $\lambda_2 > 1$ .
  - v)  $W(f) = \{\{e^{i\theta_1}\} \times \Delta\} \cup \{(e^{i\theta_1}, e^{i\theta_2})\} \cup \{\Delta \times \{e^{i\theta_2}\}\}$  iff  $f$  is of *third type*.
- Otherwise if  $f_1(x, y) = x \forall y \in \Delta$  (or respectively  $f_2(x, y) = y \forall x \in \Delta$ ) then:
- vi)  $W(f) = \Delta^2 \setminus \{\Delta \times \{e^{-i\theta_2}\}\}$  (or respectively  $W(f) = \Delta^2 \setminus \{\{e^{-i\theta_1}\} \times \Delta\}$ .)

Let  $f = (f_1, f_2) : \Delta^2 \rightarrow \Delta^2$  be holomorphic and with fixed points. By a result of Viguère (see Proposition 4.1 [12]) it follows that  $f_1(\cdot, y)$  and  $f_2(x, \cdot)$  also must have fixed points. Notice that if  $\dim \text{Fix}(f) = 2$  then  $f = id|_{\Delta^2}$ . Moreover if  $\dim \text{Fix}(f) = 1$  we know (see Propositions 2.6.10; 2.6.24 [2]) that  $\text{Fix}(f)$  is a geodesic of  $\Delta^2$  and then  $\text{Fix}(f)$  can be parametrized as  $\Delta \ni z \rightarrow (g(z), z)$  with  $g \in \text{Hol}(\Delta, \Delta)$ .

**Theorem 3.** Let  $f = (f_1, f_2) : \Delta^2 \rightarrow \Delta^2$  be a holomorphic map, not an automorphism, with fixed points in  $\Delta^2$ . Assume, up to automorphisms, that  $f(0, 0) = (0, 0)$ .

- If  $\dim \text{Fix}(f) = 0$  then  $W(f) = \emptyset$ .
- If  $\dim \text{Fix}(f) = 1$  and we let  $G := \text{Fix}(f)$  then :
  - i)  $g(z \in \text{Aut}(\Delta) \cup \{id\})$  iff  $W(f) = \partial G$  (and this is the case iff there exists a point  $(e^{i\theta}, 1) \in (\partial\Delta)^2$  which belongs to  $W(f)$ );
  - ii)  $g \notin \text{Aut}(\Delta) \cup \{id\}$  is a proper map iff  $W(f) = \emptyset$ ;

iii)  $g$  is not a proper map iff  $W(f)$  is disconnected (and this is the case iff  $f_2 = \pi_2$ ).

If  $f \in \text{Aut}(\Delta^2)$  has fixed points in  $\Delta^2$ , its components are elliptic automorphisms of  $\Delta$  and  $W(f) = \emptyset$ .

In section 2 we are going to introduce some useful tools to prove our main theorem. In particular we describe the property of the component  $f_1$  and  $f_2$  of the function  $f : \Delta^2 \rightarrow \Delta^2$  and we also study some property of the set of *Wolff points* of  $f$ . Using these results in section 3 we prove Theorem 2 and Theorem 3. Finally in section 4 we give some examples.

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## 2. PRELIMINARY RESULTS

We need now to introduce some notation and some preliminary results, as a generalization of Julia's lemma due to Abate (Abate [1]):

**Theorem 4.** (Abate [1]) *Let  $f : \Delta^n \rightarrow \Delta$  be a holomorphic map and let  $x \in \partial\Delta^n$  be such that*

$$(2.1) \quad \liminf_{\tilde{w} \rightarrow x} [k_{\Delta^n}(0, \tilde{w}) - \omega(0, f(\tilde{w}))] = \frac{1}{2} \log \alpha_f < \infty$$

*then there exists  $\tau \in \partial\Delta$  such that  $\forall R > 0$ ,  $f(E(x, R)) \subseteq E(\tau, \alpha_f R)$ . Furthermore  $f$  admits restricted  $E$ -limit  $\tau$  at  $x$ . (see (Abate [1]) for the definition).*

Consider  $f = (f_1, f_2) \in \text{Hol}(\Delta^2, \Delta^2)$ . Using the result of Hervé [see Theorem 1 in [6]] and Definition 1, we are going to examine the properties of  $f$ ,  $f_1$  and  $f_2$ .

**Lemma 5.** *Let  $f = (f_1, f_2) \in \text{Hol}(\Delta^2, \Delta^2)$ . If  $f$  is a map of first type then:*

- i) *The function  $F_1 \circ F_2$  (respectively  $F_2 \circ F_1$ ) has Wolff point  $e^{i\theta_1}$  (respectively  $e^{i\theta_2}$ ). In this case we let  $\lambda_{12}$  (respectively  $\lambda_{21}$ ) be the boundary dilatation coefficient of  $F_1 \circ F_2$  (resp.  $F_2 \circ F_1$ ) at its Wolff point;*
- ii)  *$F_1$  has non-tangential limit  $e^{i\theta_1}$  at  $e^{i\theta_2}$  and*

$$(2.2) \quad \liminf_{y \rightarrow e^{i\theta_2}} \frac{1 - |F_1(y)|}{1 - |y|} = \lambda_1 \quad ; \quad 0 < \lambda_1 < +\infty;$$

- iii)  *$F_2$  has non-tangential limit  $e^{i\theta_2}$  at  $e^{i\theta_1}$  and*

$$(2.3) \quad \liminf_{x \rightarrow e^{i\theta_1}} \frac{1 - |F_2(x)|}{1 - |x|} = \lambda_2 \quad ; \quad 0 < \lambda_2 < +\infty.$$

Furthermore  $\lambda_{12} = \lambda_{21} = \lambda_1 \lambda_2$ .

*Proof.* i). We note that if  $f$  is a map of *first type* then the function  $F_1 \circ F_2$  cannot have fixed points in  $\Delta$  or otherwise we can build fixed points in  $\Delta^2$  for  $f$ . Infact, suppose that  $x_0 \in \Delta$  is such that  $(F_1 \circ F_2)(x_0) = x_0$  and let  $F_2(x_0) = y_0$ . Then  $f(x_0, y_0) = (f_1(F_1(F_2(x_0))), F_2(x_0)), f_2(x_0, F_2(x_0))) = (x_0, y_0)$ . By Wolff's lemma,  $F_1 \circ F_2$  has a Wolff point, say  $e^{i\theta_1} \in \partial\Delta$ . Similarly  $F_2 \circ F_1$  has a Wolff point, say  $e^{i\theta_2} \in \partial\Delta$ . It is well known, by classical results of Julia, Wolff and Caratheodory [2], that:  $\liminf_{x \rightarrow e^{i\theta_1}} \frac{1 - |F_1(F_2(x))|}{1 - |x|} = \lambda_{12}$  for some  $0 < \lambda_{12} \leq 1$ ; and  $\liminf_{y \rightarrow e^{i\theta_2}} \frac{1 - |F_2(F_1(y))|}{1 - |y|} = \lambda_{21}$  for some  $0 < \lambda_{21} \leq 1$ .

ii) By Schwarz's lemma we know that  $\frac{1 - |F_1(y)|}{1 - |y|} \geq \frac{1 - |F_1(0)|}{1 + |F_1(0)|} := M$  for  $y \in \Delta$ . If we take  $y = F_2(x)$ , it implies:  $1 - |F_1(F_2(x))| \geq M(1 - |F_2(x)|)$  and dividing by  $(1 - |x|)$  we obtain:  $\frac{1 - |F_1(F_2(x))|}{1 - |x|} \geq M \frac{1 - |F_2(x)|}{1 - |x|} \geq M \frac{1 - |F_2(0)|}{1 + |F_2(0)|}$ . Let  $\lambda_2 := \liminf_{x \rightarrow e^{i\theta_1}} \frac{1 - |F_2(x)|}{1 - |x|}$  be the boundary dilatation coefficient of  $F_2$  in  $e^{i\theta_1}$ . Thus we can conclude that:  $1 \geq \lambda_{12} \geq M \lambda_2 > 0$ . Note that, since  $M \leq 1$ , it follows that  $\lambda_2$  is a finite and positive number. But we also know

that  $|F_2(x)| \rightarrow 1$  for  $x \rightarrow e^{i\theta_1}$  and by Julia's lemma we can conclude that there exists a unique  $e^{i\gamma_2}$  such that the non-tangential limit of  $F_2$  in  $e^{i\theta_1}$  is equal to  $e^{i\gamma_2}$ , that is:  $K - \lim_{x \rightarrow e^{i\theta_1}} F_2(x) = e^{i\gamma_2}$ . Let us consider the function  $(F_2 \circ F_1) : \Delta \rightarrow \Delta$ . As we proved in *i*), this function has Wolff point  $e^{i\theta_2}$ . Notice that, as for  $\lambda_2$ , one can prove that  $\lambda_1 := \liminf_{y \rightarrow e^{i\theta_2}} F_1(y)$  is such that  $0 < \lambda_1 < +\infty$ . We are going to show that:  $e^{i\gamma_2} = e^{i\theta_2}$ . Let us consider the sequence  $\{(F_2 \circ F_1)^k\}$  of iterates of  $(F_2 \circ F_1)$ . By the Wolff-Denjoy theorem it converges, uniformly on compact sets, to the constant  $e^{i\theta_2}$ . Notice that  $(F_2 \circ F_1)^k = F_2 \circ (F_1 \circ F_2)^{k-1} \circ F_1$  and, furthermore,  $e^{i\theta_1}$  is the Wolff point of  $(F_1 \circ F_2)$ . Therefore  $(F_1 \circ F_2)^k \rightarrow e^{i\theta_1}$  as  $k \rightarrow \infty$ . Let us fix  $y_0 \in \Delta$  and let  $x_0 = F_1(y_0)$ . Set  $w_k := (F_1 \circ F_2)^k(x_0)$  and  $z_k := (F_2 \circ F_1)^k(y_0)$ . Then we have  $w_k \rightarrow e^{i\theta_1}$  and  $z_k \rightarrow e^{i\theta_2}$  as  $k \rightarrow \infty$ . Moreover we notice that:  $\omega(0, w_k) - \omega(0, F_2(w_k)) = \omega(0, F_1(z_k)) - \omega(0, z_{k+1}) = \omega(0, F_1(z_k)) - \omega(0, z_k) + \omega(0, z_k) - \omega(0, z_{k+1})$ . Then

$$\begin{aligned} & - \limsup_{k \rightarrow \infty} [\omega(0, F_1(z_k))(y_0)) - \omega(0, z_k(y_0))] = \\ & \liminf_{k \rightarrow \infty} -[\omega(0, F_1(z_k))(y_0)) - \omega(0, z_k(y_0))] \\ & \geq \liminf_{w \rightarrow e^{i\theta_2}} -[\omega(0, F_1(w)) - \omega(0, w)] = \frac{1}{2} \log \lambda_1 > -\infty, \end{aligned}$$

we conclude that

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \omega(0, (F_1 \circ F_2)^k(x_0)) - \omega(0, F_2(F_1 \circ F_2)^k(x_0)) \\ & = \limsup_{k \rightarrow \infty} [\omega(0, F_1((F_2 \circ F_1)^k(y_0)) - \omega(0, (F_2 \circ F_1)^k(y_0))] < +\infty. \end{aligned}$$

Furthermore, since  $(F^2 \circ F^1)^k$  converges uniformly on compact sets to its Wolff point we have that:

$$\begin{aligned} & \limsup_{k \rightarrow \infty} [\omega(0, z_k(y_0)) - \omega(0, z_{k+1}(y_0))] = \\ & \limsup_{k \rightarrow \infty} [\omega(0, (F_2 \circ F_1)^k(y_0)) - \omega(0, (F_2 \circ F_1)^{k+1}(y_0))] \\ & \leq \limsup_{k \rightarrow \infty} \left| \frac{1 + |(F_2 \circ F_1)^k(y_0)|}{1 + |(F_2 \circ F_1)^{k+1}(y_0)|} \frac{1 - |(F_2 \circ F_1)^{k+1}(y_0)|}{1 - |(F_2 \circ F_1)^k(y_0)|} \right| \leq 0. \end{aligned}$$

We have  $\lim_{k \rightarrow \infty} F_2((F_1 \circ F_2)^k(x_0)) = e^{i\gamma_2}$  and  $e^{i\gamma_2} = \lim_{k \rightarrow \infty} F_2((F_1 \circ F_2)^k(x_0)) = \lim_{k \rightarrow \infty} (F_2 \circ F_1)^k(y_0) = e^{i\theta_2}$  thus we can conclude that  $e^{i\gamma_2} = e^{i\theta_2}$ .

*iii*). The last thing we need to prove is that  $F_1$  has non tangential limit  $e^{i\theta_1}$  at  $e^{i\theta_2}$ . This can be done proceeding as before. Using Julia-Wolff-Caratheodory theorem [14], [2] we obtain that  $\lambda_1 = \frac{\lambda_{12}}{\lambda_2}$ ,  $\lambda_2 = \frac{\lambda_{21}}{\lambda_1}$  and, in particular  $\lambda_{21} = \lambda_1 \lambda_2 = \lambda_{12}$ .  $\square$

It's clear that  $\lambda_1 > 1$  implies  $\lambda_2 < 1$  (and the converse is also true.). To simplify notations, up to automorphisms, from now on assume  $e^{i\theta_1} = e^{i\theta_2} = 1$ . Let  $\alpha_{f_1}$  = denote the number defined in theorem 4 for the function  $f_1$  at the point  $(1, 1)$  :

**Lemma 6.** *Let  $f = (f_1, f_2) : \Delta^2 \rightarrow \Delta^2$  be holomorphic and suppose that  $f_1 \neq \pi_1$  and  $f_2 \neq \pi_2$ . If  $f_1(E((1, 1), R)) \subseteq E(1, R) \forall R > 0$  then  $\alpha_{f_1} \leq 1$ .*

*Proof.* Let consider the holomorphic function  $\varphi : \Delta \rightarrow \Delta$  defined by  $\varphi(\xi) = f_1(\xi, \xi)$  and let  $\alpha_\varphi$  be the boundary dilatation coefficient of the map  $\varphi$  at the point 1. Let  $|||(z, w)||| := \max\{|z|, |w|\}$ . Since (see lemma 3.2 [1])  $\liminf_{t \rightarrow 1-} \frac{1 - |f_1(t, t)|}{1 - |t|} = \alpha_{f_1}$  then we have that  $\alpha_{f_1} = \liminf_{(z, w) \rightarrow (1, 1)} \frac{1 - |f_1(z, w)|}{1 - |||(z, w)|||} \leq \liminf_{\xi \rightarrow 1} \frac{1 - |f_1(\xi, \xi)|}{1 - |\xi|} \leq \liminf_{t \rightarrow 1-} \frac{1 - |f_1(t, t)|}{1 - |t|} = \alpha_{f_1}$ . And  $\alpha_\varphi = \liminf_{\xi \rightarrow 1} \frac{1 - |\varphi(\xi)|}{1 - |\xi|} = \liminf_{\xi \rightarrow 1} \frac{1 - |f_1(\xi, \xi)|}{1 - |\xi|} = \liminf_{t \rightarrow 1-} \frac{1 - |f_1(t, t)|}{1 - |t|} = \alpha_{f_1}$  and we can conclude that  $\alpha_\varphi = \alpha_{f_1}$ . But since  $f_1(E((1, 1), R)) \subseteq E(1, R) \forall R > 0$  then  $\alpha_\varphi \leq 1$  and this ends the proof.  $\square$

**Proposition 7.** *Let  $f = (f_1, f_2) : \Delta^2 \rightarrow \Delta^2$  be holomorphic and without fixed points in  $\Delta^2$ . Suppose that  $f_1 \neq \pi_1$  and  $f_2 \neq \pi_2$ . Fix  $R_1, R_2 > 0$  such that  $\frac{1}{R_1} = \frac{\lambda_2}{R_2}$ . If either*

- i)  *$f$  is of first type and  $\lambda_1 > 1$  or*
- ii)  *$f$  is of second type and  $\lambda_2 > 1$ ,*

Then  $f(E(1, R_1) \times E(1, R_2)) \subseteq E(1, R_1) \times E(1, R_2)$ .

*Proof.* i) Let consider  $(x_0, y_0) \in E(1, R_1) \times E(1, R_2)$ . We have:

$$\begin{aligned} \lim_{z \rightarrow 1} \omega(f_1(x_0, y_0), z) - \omega(0, z) &= \lim_{t \rightarrow 1^-} \omega(f_1(x_0, y_0), f_1(F_1(F_2(t)), F_2(t))) - \omega(0, F_1(F_2(t))) \\ &\leq \lim_{t \rightarrow 1^-} \max\{\omega(x_0, F_1(F_2(t))), \omega(y_0, F_2(t))\} - \omega(0, F_1(F_2(t))). \end{aligned}$$

If  $\max\{\omega(x_0, F_1(F_2(t))), \omega(y_0, F_2(t))\} = \omega(x_0, F_1(F_2(t)))$  then  $\lim_{t \rightarrow 1^-} \omega(x_0, F_1(F_2(t))) - \omega(0, F_1(F_2(t))) \leq \frac{1}{2} \log R_1$  because  $x_0 \in E(1, R_1)$ . If  $\max\{\omega(x_0, F_1(F_2(t))), \omega(y_0, F_2(t))\} = \omega(y_0, F_2(t))$  then:

$$\begin{aligned} &\limsup_{t \rightarrow 1^-} \omega(y_0, F_2(t)) - \omega(0, F_1(F_2(t))) \\ &\leq \limsup_{t \rightarrow 1^-} \omega(y_0, F_2(t)) - \omega(0, F_2(t)) + \limsup_{t \rightarrow 1^-} \omega(0, F_2(t)) - \omega(0, F_1(F_2(t))) \\ &\leq \frac{1}{2} \log R_2 + \limsup_{t \rightarrow 1^-} \frac{1}{2} \log \left[ \frac{1+|F_2(t)|}{1+|F_1(F_2(t))|} \frac{|1-F_1(F_2(t))|}{|1-F_2(t)|} \frac{|1-F_2(t)|}{1-t} \frac{1-t}{1-|F_2(t)|} \right] \\ &\leq \frac{1}{2} \log R_2 + \frac{1}{2} \log \lambda_1 = \frac{1}{2} \log \lambda_1 R_2 = \frac{1}{2} \log \frac{\lambda_1 R_2}{\lambda_2} = \frac{1}{2} \log R_1, \end{aligned}$$

by Julia-Wolff-Caratheodory theorem. So  $f_1(x_0, y_0) \in E(1, R_1)$ . In the same way one can prove that  $f_2(x_0, y_0) \in E(1, R_2)$ . So  $f(E(1, R_1) \times E(1, R_2)) \subseteq E(1, R_1) \times E(1, R_2)$ . This end the proof of the point i).

ii). By hypothesis it follows immediately that  $R_1 \geq R_2$  and then we have:

$$f_1(E(1, R_1) \times (E(1, R_2))) \subseteq f_1(E(1, R_1) \times \Delta) \subseteq E(1, R_1).$$

So it remains to examine the second component  $f_2$ . Again by hypothesis we have that  $\lambda_2 = \frac{R_2}{R_1}$  and we can proceed as above and we prove that  $f_2(E(1, R_1) \times (E(1, R_2))) \subseteq E(1, R_2)$  and it concludes the proof of the point ii).  $\square$

**Proposition 8.** Let  $f = (f_1, f_2) : \Delta^2 \rightarrow \Delta^2$  be holomorphic and without fixed points in  $\Delta^2$ . Suppose that  $f_1 \neq \pi_1$  and  $f_2 \neq \pi_2$ . Then  $W(f)$  is arcwise connected.

*Dim.* Let  $\Gamma_{(x,y)}$  denote the flat component of the boundary of  $\Delta^2$  containing the point  $(x, y) \in \partial\Delta^2$ . We can consider the following cases:

- 1) “ $z, w \in W(f) \cap \Gamma_{(0,1)}$ .” In this case every point of the component  $\Gamma_{(0,1)}$  is a Wolff point. Indeed, if  $x \in \Gamma_{(0,1)}$  then  $F(x, R) = E(x, R)$  and in particular they coincide with  $E(z, R) = F(z, R) = E(w, R) = F(w, R)$ . Thus if a point of the flat component  $\Gamma_{(0,1)}$  is a Wolff point then every point of this component is a Wolff point. We can conclude that there exists a continuous path, of Wolff points, that links  $z$  and  $w$ .
- 2) “ $z \in W(f) \cap \partial\Gamma_{(0,1)}$  and  $w \in W(f) \cap \Gamma_{(0,1)}$ .”  $w \in W(f)$  implies that every point of the flat component  $\Gamma_{(0,1)}$  is a Wolff point then, it is sufficient consider the radius linking  $z$  and  $w$ .
- 3) “ $z \in W(f) \cap \Gamma_{(0,1)}$  and  $w \in W(f) \cap \Gamma_{(1,0)}$ .” Every point of the flat components  $\Gamma_{(0,1)}$  and  $\Gamma_{(1,0)}$  is a Wolff point of  $f$ . By definition of Wolff points and horosphere, it follows that:  $f(E(1, R) \times E(1, R)) = f((\Delta \times E(1, R)) \cap (E(1, R) \times \Delta)) \subseteq E(1, R) \times E(1, R) \forall R > 0$  and  $(1, 1) \in W(f)$ ; thus there is a continuous path that links  $z$  and  $w$ , passing through the point  $(1, 1)$ .

We are now going to prove that there isn't any other possibility. In fact the following cases are not possible:

- a) “ $\exists z \in W(f) \cap \Gamma_{(0,1)}$  e  $\exists w \in W(f) \cap \Gamma_{(0,-1)}$ .” In this case, by definition of Wolff point, we would have:  $f_2(\Delta \times E(1, R)) \subseteq E(1, R) \forall R > 0$  and  $f_2(\Delta \times E(-1, R)) \subseteq E(-1, R) \forall R > 0$ . And if we consider  $y_0 \in E(1, R_1) \cap E(-1, R_2)$  and  $x \in \Delta$  we would have:  $f_2(x, y_0) = y_0 \forall x \in \Delta$ . But it is possible only if  $f_2(x, y) = y$  for all  $x \in \Delta$  and it isn't our case.
- b) “ $\exists z \in W(f) \cap \partial\Gamma_{(0,1)}$  and  $\exists w \in W(f) \cap \partial\Gamma_{(0,1)}$ ,  $z \neq w$ .” We can suppose that  $z = (-1, 1)$  and  $w = (1, 1)$ . Then  $f_2(\Delta \times E(1, R)) \subseteq E(1, R) \forall R > 0$  and  $f_2(\Delta \times$

$E(-1, R) \subseteq E(-1, R) \forall R > 0$ . Again, if we take  $y_0 \in \partial E(1, R_1) \cap \partial E(-1, R_2)$ , chosen opportunely  $R_1, R_2 > 0$ , we obtain that  $f_2(x, y_0) = y_0 \forall x \in \Delta$ . But it is possible only if  $f_2(x, y) = y \forall x \in \Delta$  and it isn't our case.

c) “ $(1, 1); (-1, -1) \in W(f)$ ”. In this case we have  $f(E(-1, R) \times E(-1, R)) \subseteq E(-1, R) \times E(-1, R) \forall R > 0$  and  $f(E(1, R) \times E(1, R)) \subseteq E(1, R) \times E(1, R) \forall R > 0$ . Then if we take  $0 \in \partial E(1, R) \cap \partial E(-1, R)$  we obtain that  $(0, 0) \in [\partial(E(-1, R) \times E(-1, R))] \cap [\partial E(1, R) \times E(1, R)]$  and consequently:  $f(0, 0) = 0$ , but it is not possible because we supposed  $f$  without fixed points in the bidisc.  $\square$

### 3. WOLFF POINTS

We are finally ready to prove our main result, theorem 2, taking  $e^{i\theta_1} = e^{i\theta_2} = 1$ :

*Proof of Theorem 2.* We are going to prove every statement in the direction “ $\Leftarrow$ ”.

ii). Let apply Lemma 4 to the holomorphic function  $f_1 : \Delta^2 \rightarrow \Delta$  and let us study:

$$(3.1) \quad \liminf_{(x,y) \rightarrow (1,1)} K_{\Delta^2}((0,0), (x,y)) - \omega(0, f_1(x,y))$$

Consider the direction  $x = F_1(w)$   $y = w$ . We obtain:  $\omega(0, x) = \omega(0, F_1(w))$  and  $\omega(0, y) = \omega(0, w)$ . Since  $\lambda_1 \leq 1$ , then:  $\liminf_{w \rightarrow 1} [\omega(0, w) - \omega(0, F_1(w))] \leq 0$ . In particular there exists a subsequences  $w_k$  such that  $\lim_{k \rightarrow \infty} w_k = 1$  and  $\lim_{k \rightarrow +\infty} [\omega(0, w_k) - \omega(0, F_1(w_k))] \leq 0$ . If we look at (3.1) along the direction  $w_k$  we obtain:

$$\begin{aligned} & \liminf_{(x,y) \rightarrow (1,1)} K_{\Delta^2}((0,0), (x,y)) - \omega(0, f_1(x,y)) \leq \\ & \liminf_{k \rightarrow +\infty} K_{\Delta^2}((0,0), (F_1(w_k), w_k)) - \omega(0, f_1(F_1(w_k), w_k)) = \\ & = \liminf_{k \rightarrow +\infty} \omega(0, F_1(w_k)) - \omega(0, F_1(w_k)) = 0. \end{aligned}$$

Then, by (3.1), there exists  $\tau_1 \in \partial \Delta$  such that  $f_1(E(1, 1), R) \subseteq E(\tau_1, R) \forall R > 0$ . In particular  $f_1$  admits restricted E-limit  $\tau_1$ . Let apply again the 3.1 to the map  $f_2 : \Delta^2 \rightarrow \Delta$  along the direction  $x = z$ ;  $y = F_2(z)$ . Proceeding as above, we obtain  $\liminf_{(x,y) \rightarrow (1,1)} K_{\Delta^2}((0,0), (x,y)) - \omega(0, f_2(x,y)) = 0$ . Then there exists  $\tau_2 \in \partial \Delta$  such that  $f_2(E(1, 1), R) \subseteq E(\tau_2, R) \forall R > 0$  and  $f_2$  admits restricted E-limit  $\tau_2$ . We claim that  $\tau_1 = \tau_2 = 1$ . Consider the curves  $\sigma_1(t) = (F_1(t), t)$  and  $\sigma_2(t) = (t, F_2(t))$ ;  $t \in [0, 1]$ . These curves,  $\sigma_1, \sigma_2$ , are peculiar  $(1, 1)$ -curves. Furthermore  $f_1(\sigma_1(t)) = f_1(F_1(t), t) = F_1(t) \rightarrow 1$  as  $t \rightarrow 1^-$  and  $f_2(\sigma_2(t)) = f_2(t, F_2(t)) = F_2(t) \rightarrow 1$  as  $t \rightarrow 1^-$ . Since  $\tau_1, \tau_2$ , are respectively restricted E-limit of  $f_1$  and restricted E-limit of  $f_2$  then we can conclude that  $\tau_1 = 1$  and  $f_1(E((1, 1), R)) \subseteq E(1, R) \forall R > 0$ ;  $\tau_2 = 1$  and  $f_2(E((1, 1), R)) \subseteq E(1, R) \forall R > 0$ . Thus if  $f$  is of *first type* and  $\lambda_1 \leq 1, \lambda_2 \leq 1$  then:

$$f((E(1, 1), R) = (f_1(E((1, 1), R)), f_2(E((1, 1), R))) \subseteq E((1, 1), R) \forall R > 0$$

and in particular  $\tau = (1, 1) \in W(f)$ . Notice that, by the proof of Lemma 8, points as  $(-1, 1)$  and  $(1, -1)$  cannot be Wolff point of  $f$ , because they are on the Silov boundary of the same flat component of the point  $(1, 1)$ . Furthermore, also, no points of the flat components  $\Gamma_{(0,1)}$ ,  $\Gamma_{(1,0)}$  can be Wolff points of  $f$  otherwise, by definition, the point 1 would be a Wolff point for  $f_1(\cdot, y)$  and  $f_2(x, \cdot)$  and it isn't possible because  $f_1(\cdot, y)$  has fixed points described by the function  $F_1$  and  $f_2(x, \cdot)$  has fixed points described by the function  $F_2$ . By lemma 8, we know that  $W(f)$  is arcwise connected, then the unique Wolff point of  $f$  must be  $(1, 1)$ .

i) Since  $F_1$  is a holomorphic self-map of  $\Delta$ , there are at most two possibilities:  $F_1(y_0) = y_0$  for some  $y_0 \in \Delta$  or  $F_1$  has a Wolff point  $\tau_1 \neq 1$ . We can suppose,  $\tau_1 = -1$ . In this last case we have  $K - \lim_{y \rightarrow -1} F_1(y) = -1$  and the angular derivative of  $F_1$  in  $\{-1\}$  is  $\delta_1 \leq 1$ . Suppose  $F_1$  has a fixed point. Since  $\lambda_1 > 1$  then  $\lim_{w \rightarrow 1} \omega(0, w) - \omega(0, F_1(w)) > 0$ . Proceeding as before we obtain:

$$\begin{aligned} & \liminf_{(x,y) \rightarrow (1,1)} K_{\Delta^2}((0,0), (x,y)) - \omega(0, f_1(x,y)) \\ & \leq \liminf_{(x,y) \rightarrow (1,1)} K_{\Delta^2}((0,0), (F_1(w), w)) - \omega(0, F_1(w)) = \frac{1}{2} \log \lambda_1 < +\infty \end{aligned}$$

and by Julia's lemma for polydiscs  $f_1$  admits restricted E-limit  $\tau_1 = 1$ . Then the inferior limit (3.1) is bounded. Let  $\alpha$  denote this inferior limit. This is a sort of "boundary dilatation coefficient" of  $f_1$  in  $(1, 1)$ . We can write:

$$\begin{aligned} +\infty & > \frac{1}{2} \log \alpha = \liminf_{(x,y) \rightarrow (1,1)} K_{\Delta^2}((0,0), (x,y)) - \omega(0, f_1(x,y)) = \\ & = \frac{1}{2} \log \liminf_{(x,y) \rightarrow (1,1)} \frac{1 - |f_1(x,y)|}{1 - |||(x,y)|||} = \frac{1}{2} \log \liminf_{t \rightarrow 1^-} \frac{1 - |f_1(t,t)|}{1 - t}. \end{aligned}$$

Let consider the holomorphic function  $\varphi : \Delta \rightarrow \Delta$  defined by  $\varphi(\xi) = f_1(\xi, \xi)$ . By Lemma 6,  $\alpha_\varphi = \alpha_{f_1}$ . If  $F_1(y_0) = y_0$  it implies that  $\varphi(y_0) = f_1(F_1(y_0), y_0) = F_1(y_0) = y_0$  then  $\varphi(\xi)$  has a fixed point in  $\Delta$ . We also know that  $\sigma(t) = (t, t)$  is a peculiar  $(1, 1)$ -curve then  $f_1(t, t) \rightarrow 1$  when  $t \rightarrow 1^-$ . It follows that the point 1 is a fixed point of  $\varphi$  on the boundary of  $\Delta$  and since  $\varphi$  has a fixed point then it must be  $\alpha_{f_1} = \alpha_\varphi > 1$ . By Lemma 6, this condition is sufficient to say that the point  $(1, 1)$  cannot be a Wolff point for the map  $f$ . In fact we have:  $f_1(E((1, 1), R)) \subseteq E(1, \alpha_{f_1} R) \quad \forall R > 0$  with  $\alpha_{f_1} > 1$ . It implies  $f_1(E((1, 1), R)) \not\subseteq E(1, R)$  and the point  $(1, 1)$  cannot be a Wolff point of  $f$ .

Suppose, now,  $F_1$  has no fixed points. Then  $\liminf_{(x,y) \rightarrow (-1,-1)} \max\{\omega(0, x), \omega(0, y)\} - \omega(0, f_1(x, y)) \leq \liminf_{w \rightarrow -1} \omega(0, F_1(w)) - \omega(0, F_1(w)) = 0$ , and then by lemma 3.1,  $f_1$  has restricted E-limit, say  $\tau \in \partial\Delta$ , in  $(-1, -1)$ . Furthermore  $\sigma_1(t) = (F_1(t), t)$  is a peculiar  $(-1, -1)$ -curve and then, proceeding as before we have that  $\tau = -1$  and  $f_1(E(-1, R) \times E(-1, R)) \subseteq E(-1, R) \quad \forall R > 0$ . If the point  $(1, 1)$  is a Wolff point of  $f$  we have:  $f_1(E(1, R) \times E(1, R)) \subseteq E(1, R) \quad \forall R > 0$ , and then, chosen  $R > 0$  such that  $\{0\} \in \partial E(1, R) \cap \partial E(-1, R)$  we will have  $(0, 0) \in \partial E((1, 1), R) \cap \partial E((-1, -1), R)$  and thus  $f_1(0, 0) \in E(1, R) \cap E(-1, R) = \{0\} \Rightarrow f_1(0, 0) = 0 \Rightarrow F_1(0) = 0$ . But it isn't possible because we supposed that  $F_1$  has no fixed points in  $\Delta$ . So, also in case  $i$ ) the point  $(1, 1)$  cannot be a Wolff point of  $f$ . We can note, now, that the flat component of the boundary cannot be a Wolff component of  $f$  for otherwise  $f_1(\cdot, y)$  or  $f_2(x, \cdot)$  would have Wolff points but it isn't possible because, in this case, they have, both, fixed points in  $\Delta$ . By Lemma 8 we know that two points on the Silov boundary of the same flat component cannot be, at the same time, Wolff point of the map  $f$ . Furthermore  $W(f)$  is arcwise connected, so we have just one more possibility: there is one point (different from  $(1, 1)$ ) on the Silov boundary that is the Wolff point of  $f$ . But:

- the point  $(1, -1)$  cannot be Wolff points of  $f$  because otherwise we would have  $f_2(E(1, R) \times E(1, R)) \subseteq E(1, R) \quad \forall R > 0$  since  $\lambda_2 \leq 1$  and  $f_2(E(1, R) \times E(-1, R)) \subseteq E(-1, R) \quad \forall R > 0$ .

Chosen  $0 \in \partial E(-1, R) \cap E(1, R)$  and  $x \in E(1, R)$  then  $(x, 0) \in [E((1, 1), R)] \cap [E((-1, 1), R)]$  and we obtain  $f_2(x, 0) = 0 \quad \forall x \in E(1, R)$  and then  $F_2(x) \equiv 0$  but it is inconsistent with our hypothesis. So the point  $(1, -1)$  cannot be Wolff points of  $f$ .

- the point  $(-1, -1)$  cannot be Wolff point of  $f$  because otherwise  $f_2(E(-1, R) \times E(-1, R)) \subseteq E(-1, R) \quad \forall R > 0$ . As above, chosen  $0 \in \partial E(-1, R) \cap E(1, R)$  we have  $f_2(0, 0) = 0$  that implies  $F_2(0) = 0$  but it isn't possible because  $F_2$  has Wolff point 1.

- the point  $(-1, 1)$  cannot be Wolff point of  $f$ . This last statement follows by the point  $i$ ) of Proposition 7 Indeed if  $(-1, 1)$  were a Wolff point, we would have:  $f(E(-1, R) \times E(1, R)) \subseteq E(-1, R) \times E(1, R) \quad \forall R > 0$ . In particular, chosen  $R_1, R_2$  such that  $\frac{\lambda_{12}}{R_1} = \frac{\lambda_2}{R_2}$ ,  $\exists x_0 \in \partial E(-1, R) \cap \partial E(1, R_1)$  such that  $f(x_0, y) = (x_0, \tilde{y})$  for all  $y \in E(1, R_2)$  and also such that  $f_1(x_0, y) = x_0 \quad \forall y \in E(1, R_2)$ . It implies that  $F_1(y) = x_0 \quad \forall y \in E(1, R_2)$ . But it isn't possible. So we proved that there isn't any Wolff point for  $f$  and in this way we end the proof of the point  $i$ ) of the theorem.

iii), iv). If  $f$  is of *second type* it is clear that every point of the flat component  $\Gamma_{(1,0)} = \{1\} \times \Delta$  is a Wolff point, in fact, chosen  $(1, \tilde{y}) \in \Gamma_{(1,0)}$ , the small horosphere centered in this point is  $E((1, \tilde{y}), R) = E(1, R) \times \Delta \ \forall \tilde{y} \in \Delta$  and we have:

$$f(E((1, \tilde{y}), R)) = (f_1(E(1, R) \times \Delta), f_2(E(1, R) \times \Delta))$$

but  $f_1(E(1, R) \times \Delta) \subseteq E(1, R)$  because  $\tau = 1$  is the Wolff point of  $f_1(\cdot, y) \ \forall y \in \Delta$  and  $f_2(E(1, R) \times \Delta) \subseteq \Delta$ . It follows that

$$f(E((1, \tilde{y}), R)) \subseteq E(1, R) \times \Delta = E((1, \tilde{y}), R) \ \forall \tilde{y} \in \Delta$$

and then we have that every point of the flat component  $\Gamma_{(1,0)}$  is a Wolff point of  $f$ . Furthermore, applying again Julia's lemma for polydiscs, as in the proof of point ii) of this theorem, we have that  $\lambda_2 \leq 1$  implies  $f_1(E((1, 1), R)) \subseteq E((1, 1), R) \ \forall R > 0$  and  $\{(1, 1)\} \in W(f)$ . If on the other hand  $\lambda_2 > 1$ , by the proof of the points i) and ii) of the theorem, we have that  $\{(1, 1)\} \notin W(f)$ . Also in these cases we need to prove that there isn't any other Wolff point. If  $\lambda_2 < 1$  the points of the flat component  $\Delta \times \{1\}$  cannot be Wolff points for  $f$  because  $f_2$  has fixed points. Moreover by lemma 8, the points of the Silov boundary of  $\Gamma_{01}$  cannot be Wolff point because  $\{(1, 1)\} \in W(f)$ . Then since  $W(f)$  is arcwise connected there isn't any other possibility and the set of the Wolff points of  $f$  must be  $W(f) = \{\{1\} \times \Delta\} \cup \{(1, 1)\}$ . If  $\lambda_2 > 1$ , neither the points of the flat component  $\Gamma_{(0,1)}$  nor the points of  $\Gamma_{(0,-1)}$  can be Wolff points because  $f_2$  has fixed points. Furthermore we have  $\{(1, -1)\} \notin W(f)$ . Indeed  $f_1(E(1, R) \times \Delta) \subseteq E(1, R) \ \forall R > 0$  and  $f_2(E(1, R) \times E(1, R)) \subseteq E(1, R) \ \forall R > 0$ . Thus, using Proposition 7, we can prove that  $\{(1, -1)\} \notin W(f)$ .

v) As in iii) and iv), we have that every point of the flat components of the boundary,  $\{1\} \times \Delta$  and  $\Delta \times \{1\}$ , is a Wolff point of  $f$ . It implies  $\{(1, 1)\} \in W(f)$ , indeed  $f(E((1, 1), R)) = (f_1([E(1, R) \times \Delta] \cap [\Delta \times E(1, R)]), f_2([E(1, R) \times \Delta] \cap [\Delta \times E(1, R)])) \subseteq E(1, R) \times E(1, R) = E((1, 1), R)$ . Thus  $W(f) \neq \emptyset$  and it is at least  $[\{1\} \times \Delta] \cup \{(1, 1)\} \cup [\Delta \times \{1\}]$ . By lemma 8, it follows that it must be exactly  $W(f) = [\{1\} \times \Delta] \cup \{(1, 1)\} \cup [\Delta \times \{1\}]$ . In this way we proved every statement of the theorem in the direction  $\Leftarrow$ . Then we can conclude that also the implications in direction  $\Rightarrow$  are proved.

vi) Let suppose that  $f_1(x, y) = x \ \forall y \in \Delta$  and let examine the set  $W(f)$ . Let note that  $f_2 \neq \pi_2$  otherwise  $f = id_\Delta$ . Furthermore there is no a holomorphic map  $F_2 : \Delta \rightarrow \Delta$  such that  $f_2(x, F_2(x)) = F_2(x) \ \forall x \in \Delta$ . Otherwise we will have that  $\exists x_0 \in \Delta$  such that  $f(x_0, F_2(x_0)) = (f_1(x_0, F_2(x_0)), f_2(x_0, F_2(x_0))) = (x_0, F_2(x_0))$  and it is not possible because we supposed  $f$  without fixed points in the bidisc. Then, by Hervé theorem,  $f_2$  has Wolff point  $\tau_{f_2} = e^{i\theta_2}$ . Let suppose, without loss of generality, that  $e^{i\theta_2} = 1$ . Then every point of  $\Gamma_{(0,1)}$  is a Wolff point of  $f$ , indeed, by  $f_1 = id$  and  $\tau_{f_2} = 1$  we obtain:  $f(\Delta \times E(1, R)) = (f_1(\Delta \times E(1, R)), f_2(\Delta \times E(1, R))) \subseteq \Delta \times E(1, R) \ \forall R > 0$  and, also:  $f(E(1, R) \times \Delta) = (f_1(E(1, R) \times \Delta), f_2(E(1, R) \times \Delta)) \subseteq E(1, R) \times \Delta \ \forall R > 0$ . It follows immediately that every point of  $\Gamma_{(1,0)}$  is a Wolff point and then, the point  $\{(1, 1)\}$  is a Wolff point of  $f$ . In the same way we can prove that  $\Gamma_{(-1,0)}$  is a Wolff component of  $f$ , and then also the point  $\{(-1, 1)\} \in W(f)$ . Thus  $f$  fixes every leaf  $(k, y)$ :  $f(k, y) = (f_1(k, y), f_2(k, y)) = (k, \tilde{y})$  and in particular  $y \in E(1, R)$  implies  $\tilde{y} \in E(1, R)$ . By this remark follows that the points of the flat component of the boundary  $\Gamma_{(0,-1)} = \Delta \times \{-1\}$  cannot be Wolff points for  $f$  otherwise we will have:  $f_2(\Delta \times E(1, R)) \subseteq E(1, R) \ \forall R > 0$  and  $f_2(\Delta \times E(-1, R)) \subseteq E(-1, R) \ \forall R > 0$ . Then chosen  $\{0\} \in \partial E(1, R) \cap \partial E(-1, R)$  we have  $f_2(\Delta \times E(1, R)) \subseteq [E(1, R) \cap E(-1, R)] = \{0\}$  and;  $f_2(x, 0) = x \ \forall x \in \Delta$ . Consequently:  $f(0, 0) = (f_1(0, 0), f_2(0, 0)) = (0, 0)$  but it is inconsistent with the hypothesis that  $f$  has not fixed point in  $\Delta^2$ . Also the points  $\{(-1, -1)\}$  e  $\{(1, -1)\}$  cannot be Wolff points of  $f$ , because we already know that  $\{(1, 1)\}$  and  $\{(-1, 1)\}$  are Wolff points



of  $f$ . In fact if  $\{(-1, -1)\} \in W(f)$  were we would have:  $f_2(E(-1, R) \times E(-1, R)) \subseteq E(-1, R) \forall R > 0$  and  $f_2(E(1, R) \times E(1, R)) \subseteq E(1, R) \forall R > 0$ . As done before, we can choose  $\{0\} \in \partial E(1, R) \cap \partial E(-1, R)$  and we obtain:  $f(0, 0) = (f_1(0, 0), f_2(0, 0)) = (0, 0)$  but it isn't possible. In the same way we can prove that  $\{(-1, 1)\} \notin W(f)$ . Then:  $W(f) = \{\{-1\} \times \Delta\} \cup \{(-1, 1)\} \cup \{\Delta \times \{1\}\} \cup \{(1, 1)\} \cup \{\{1\} \times \Delta\}$  and it ends the proof of the theorem.  $\square$

We can end this section proving theorem 3. Recall that by hypothesis  $f_i(0, 0) = (0, 0)$  and  $F_i(0) = 0, i = 1, 2$ . Moreover if we denote by  $K((0, 0), R)$  the Kobayashi disk centered in  $(0, 0)$  with radius  $R$ , we have that  $f(K((0, 0), R)) \subseteq K((0, 0), R)$  for all  $R > 0$ .

*Proof. Theorem 3.*

*Remark 9.* Let denote by  $\Gamma_{(x, y)}$  the flat component of  $\partial\Delta^2$ , containing the point  $(x, y)$ .

Recall that by hypothesis  $f_i(0, 0) = (0, 0)$  and  $F_i(0) = 0, i = 1, 2$ . Moreover if we denote by  $K((0, 0), R)$  the Kobayashi disk centered in  $(0, 0)$  with radius  $R$ , we have that  $f(K((0, 0), R)) \subseteq K((0, 0), R)$  for all  $R > 0$ .

Suppose first that  $dmFix(f) = 0$ . If a point of the Silov boundary, say  $(1, 1)$ , is a Wolff point then  $f(E(1, 1), R) \subseteq (E(1, 1), R)$  for all  $R > 0$ . If we take  $(0, 0) \neq (x_0, y_0) = [\partial(E(1, 1), R) \cap \partial K((0, 0), R_1)]$  we have that  $(x_0, y_0)$  must be fixed by  $f$  and it is a contradiction. On the other hand if a point of a flat component, say  $(0, 1)$ , is a Wolff point, every point of that flat component is a Wolff point for  $f$ . In this case we have  $f(\Delta \times E(1, R)) \subseteq \Delta \times E(1, R)$  for all  $R > 0$ . Chosen  $(x, y_0) \in [\partial\Delta \times E(1, R)] \cap \partial K((0, 0), R)$  we obtain that  $f(x, y_0) = (x_1, y_0)$  for every  $x \in \Delta$ . Then  $f_2(x, y) = y$  but it is a contradiction because, in this case  $\dim Fix(f) = 0$  and also  $\dim Fix f_2 = 0$ . Thus we conclude that  $W(f) = \emptyset$ .

Now, suppose that  $\dim Fix f = 1$ .

1) If  $(e^i\theta, 1) \in W(f)$  then  $f(E((e^i\theta, 1), R)) \subseteq E((e^i\theta, 1), R)$  for all  $R > 0$ . Since, by definition,  $K((0, 0), R_1)$  is the product of two Poincaré discs of radius  $R_1$ , we can take  $(0, 0) \neq (e^i\theta x_0, x_0) = [\partial(E(e^i\theta, 1), R) \cap \partial K((0, 0), R_1)]$ . We have that  $(e^i\theta x_0, x_0)$  must be fixed by  $f$ . We can do the same thing for every point  $(e^i\theta x, x) \in \Delta^2$ , changing the radius  $R_1$ . Then  $Fix(f)$  is equal to the geodesic  $(e^i\theta z, z)$ . Using the estimate of the inferior limit in lemma 4 as in the points of theorem refW(f) we see that  $\partial G = W(f)$ . Indeed any other point of the flat component of  $\partial\Delta^2$  cannot be a Wolff point since  $f_i, i = 1, 2$  has fixed points. On the other hand if  $g(z) = e^i\theta z$  then  $(e^i\theta, 1) \in W(f)$  and every point of the boundary of  $\partial G \in W(f)$ . To prove this fact it is sufficient to apply lemma 4 and study the inferior limit 2.1 along the direction  $(e^i\theta t, t)$ , as in the proof of the theorem 2.

2), 3) Suppose that  $g$  is proper and  $g$  is neither an automorphism nor the identity. Then  $W(f)$  is contained in the Silov boundary of  $\Delta^2$  and it is in contradiction with the preceding point. Then  $W(f) = \emptyset$ . It proves the implication " $\Rightarrow$ " of point 2. Suppose now that  $g$  is not a proper map and let prove that  $W(f) \neq \emptyset$  and it is disconnected. Since  $g$  is not proper then there exists a sequence  $z_k \in \Delta$  such that  $z_k \rightarrow e^{i\theta} \in \partial\Delta$  as  $k \rightarrow \infty$  and  $g(z_k) \rightarrow c \in \Delta$  as  $k \rightarrow \infty$ . By theorem 4 we get that

$$\begin{aligned} & \liminf_{(x, y) \rightarrow (e^{i\theta}, c)} K_{\Delta^2}((0, 0), (x, y)) - \omega(0, f_i(x, y)) \leq \\ & \leq \liminf_{k \rightarrow \infty} K_{\Delta^2}((0, 0), (g(z_k), z_k)) - \omega(0, f_i(g(z_k), z_k)) = \\ & \liminf_{k \rightarrow \infty} \max\{\omega(0, g(z_k)); \omega(0, z_k)\} - \omega(0, f_i(g(z_k), z_k)) = (\star) \end{aligned}$$

if  $i = 1$  then  $(\star) = \infty$ ; and if  $i = 2$  then  $(\star) = 0$ .

Thus we get

$$f(E(e^{i\theta}, c), R) \subseteq (E(e^{i\theta}, c), R) \forall R > 0$$

and  $(e^{i\theta}, c) \in W(f) \neq \emptyset$ . Thus  $\Gamma(e^{i\theta}, c) \in W(f)$ . It means that  $f_2(x, \cdot)$  has Wolff point but since  $f_2$  has also fixed points it follows that  $f_2 = \pi_2$ . It follows also that the flat component of the boundary  $\Gamma(e^{-i\theta}, c) \in W(f)$ . Any other point of the flat component can be a Wolff point since  $f_1 \neq \pi_1$  and  $f_1(\cdot, y)$  has not Wolff points. Any other point of the Silov boundary can be a Wolff point since the point 1) of the theorem holds. Thus  $W(f)$  is disconnected. In this way we proved implication  $\Rightarrow$  of point 3). Now we have that if  $W(f) = \emptyset$  then  $g$  is proper for the preceding point. And if  $f_2 = \pi_2$  then  $g$  is not proper since  $W(f) \neq id$ .  $\square$

#### 4. EXAMPLES

We can now give an example for each case of the theorem:

*Example i) of theorem 2* We are going to give an example of a holomorphic self map  $f$ , of the complex bidisc, without fixed points and without Wolff points. Let consider:  $f(x, y) = (\frac{1}{2}(x + F_1(y)), \frac{1}{2}(y + F_2(x)))$  with  $F_1$  and  $F_2$  holomorphic self map of the unit disc  $\Delta$ . We can choose  $F_1$  and  $F_2$  such that  $f$  hasn't fixed points in  $\Delta^2$  that is for example:  $F_1(y) = y^2$  and  $F_2(x) = \frac{3x+1}{x+3}$ . We are going to prove that the point  $(1, 1)$  isn't Wolff point for  $f$ . By lemma 6 it will be sufficient prove that the limit ( 2.1) is strictly greater than 0. If  $\max\{\omega(0, x), \omega(0, y)\} = \omega(0, x)$  :

$$\begin{aligned} \liminf_{(x,y) \rightarrow (1,1)} \max\{\omega(0, x), \omega(0, y)\} - \omega(0, f_1(x, y)) &= \liminf_{(x,y) \rightarrow (1,1)} \omega(0, x) - \omega(0, \frac{1}{2}(x + F_1(y))) \\ &\geq \liminf_{(x,y) \rightarrow (1,1)} \frac{1}{2} \log \left[ \frac{1+|x|}{1+|\frac{1}{2}(x+F_1(y))|} \frac{1-|\frac{1}{2}x| - |\frac{1}{2}F_1(y)|}{1-|x|} \right] > \\ &\liminf_{(x,y) \rightarrow (1,1)} \frac{1}{2} \log \left[ \frac{1+|x|}{1+|\frac{1}{2}(x+F_1(y))|} \frac{1-|\frac{1}{2}x| - |\frac{1}{2}x|}{1-|x|} \right] = 0. \end{aligned}$$

Then the boundary dilatation coefficient of  $f_1$  at the point  $z = 1$  is  $\alpha_{f_1} > 1$ . On the other hand if  $\max\{\omega(0, x), \omega(0, y)\} = \omega(0, y)$  we have:

$$\begin{aligned} \liminf_{(x,y) \rightarrow (1,1)} \max\{\omega(0, x), \omega(0, y)\} - \omega(0, f_1(x, y)) &= \liminf_{(x,y) \rightarrow (1,1)} \omega(0, y) - \omega(0, \frac{1}{2}(x + F_1(y))) \\ &\geq \liminf_{(x,y) \rightarrow (1,1)} \frac{1}{2} \log \left[ \frac{1+|y|}{1+|\frac{1}{2}(x+F_1(y))|} \frac{1-|\frac{1}{2}x| - |\frac{1}{2}F_1(y)|}{1-|y|} \right] \\ &> \liminf_{(x,y) \rightarrow (1,1)} \frac{1}{2} \log \left[ \frac{1+|y|}{1+|\frac{1}{2}(x+F_1(y))|} \frac{1-|\frac{1}{2}y| - |\frac{1}{2}y|}{1-|y|} \right] = 0. \end{aligned}$$

Then  $\forall R > 0$   $f_1(E((1, 1), R)) \subseteq E(1, \alpha_{f_1} R)$  with  $\alpha_{f_1} > 1$  and consequently, by the previous proof  $(1, 1)$  cannot be a Wolff point for  $f$ . Furthermore, by the proof of *i)* we can conclude that there isn't any other Wolff point.

*Example ii) of theorem 2:*  $f(x, y) = (\frac{1}{2}(x + \frac{5y+3}{3y+5}), \frac{1}{2}(y + \frac{3x+1}{x+3}))$ .

*Example iii) of theorem 2:*  $f(x, y) = (\frac{3x+1}{x+3}, \frac{1}{2}(y + F_2(x)))$  with  $F_2(x) = \frac{5x+3}{3x+5}$ .

*Example iv) of theorem 2:*  $f(x, y) = (\frac{3x+1}{x+3}, \frac{1}{2}(y + F_2(x)))$  with  $F_2(x) = x^2$ .

*Example v) of theorem 2*  $f(x, y) = (\frac{3x+1}{x+3}, \frac{5y+3}{3y+5})$ .

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